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# Solving Sphere Packing Problem by DC Local Search Method

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**ABSTRACT**
This paper addresses a nonconvex optimization problem, the packing problem with inequality constraints simulation on DC (Difference of Convex) functions. We formulate a general class of packing problems as DC programming and use DC local search techniques. The DC local search method leverages the DC decomposition of the objective and constraint functions, allowing for iterative refinement through convex sub-problems. Each iteration involves solving a convex approximation of the original problem, guided by sub-differential information to ensure descent and feasibility. We demonstrate the effectiveness of this approach through various geometric packing scenarios, especially circle packing, showing significant improvements in packing density and computational efficiency. The DC local search method provides a flexible and scalable framework for tackling complex packing configurations in two dimensions. In test problem, we used Sangaku optimization problem, the packing 6 circle into rectangle of 1:1.934798 size. Computational results were obtained using Python in a Jupyter Notebook.

**KEYWORDS**
Packing problems, Difference of Convex functions, Local search method

# Introduction

Packing problems constitute a fundamental class of optimization problems that involve the efficient arrangement of objects within a given space, subject to constraints such as non-overlapping, containment, and spatial boundaries. The general objective is to arrange a set of objects—such as circles, spheres, or boxes—within a container in a way that avoids overlaps while optimizing criteria such as maximizing packing density or minimizing container size. Due to their inherently nonconvex and combinatorial nature, packing problems are typically NP-hard and pose significant challenges for traditional optimization methods. DC (Difference of Convex) programming provides a powerful framework for addressing such nonconvex optimization problems by

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decomposing them into the difference of two convex functions. In the mid-1990s, Pham Dinh Tao and Le Thi Hoai An introduced the DC Algorithm (DCA), a method that solves DC optimization problems by iteratively refining solutions through convex sub-problems [1]. The algorithm is designed to converge to local optima by linearizing the concave part of the objective function. In recent years, significant contributions have been made by A. S. Strekalovsky, who has advanced DC programming by bridging it with global and local search algorithms, exact penalization theory, and global optimality conditions [2]-[6]. His work strengthens both the theoretical footing and practical performance of DC local search methods in constrained, non-convex settings. In this paper, we proposes a on formulating for a class of packing problems within the DC (Difference of Convex) form ,where the objective and constraints are reformulated into DC forms and solving it using the DC Local Search Method. Many components of packing problems are nonconvex due to the geometric constraints, such as distance constraints, collision avoidance, and shape interactions—can be formulated or approximated using DC functions. Using a local search method based on the DC, we develop an iterative algorithm that efficiently solves. Under suitable conditions, we analyze the convergence of the method using tools such as the Kurdyka–Łojasiewicz (KL) property, which guarantees global convergence to critical points. The DC Local Search Method emerges as a tailored strategy for solving packing problems. It leverages the DC decomposition of the objective and constraint functions and iteratively searches for better local solutions via convex optimization sub-problems. Finally, we demonstrate how the proposed method effectively navigates the nonconvex search space and produces high-quality solutions by exploiting problem structure, with implementations carried out using the GEKKO package.

The rest of the paper is organized as follows: Section 2 reviews general formulation on packing problem. Section 3 formulates DC programming and DC Local search method and DC reformulation on the packing problem. Section 4 discusses the convergence of the DC local search algorithm and provides its theoretical analysis. Section 5 presents the test problem and corresponding computational results. Section 6 concludes the paper and outlines directions for future research.

## 1. General formulation of packing problem

Define a sphere with radius $r$ and center $c^0 \in \mathbb{R}^n$.

$$B(c^0, r) = \{x \in \mathbb{R}^n | \|c^0 - x\| \leq r\}.$$

and let $D \subset R^n$ is a bounded and polyhedral nonempty set :

$$D = \{x \in \mathbb{R}^n | \langle a^i, x\rangle \leq b_i, \;\; a^i \in \mathbb{R}^n, \;\; b_i \in \mathbb{R}, \;\; i = 1, \dots, m\}.$$

**Theorem 1.1.** *$B(c^0, r) \subset D$ if and only if*

$$\langle a^i, x\rangle + c^0\|a^i\| \leq b_i, \;\; i = 1, \dots, m.$$

***Proof.*** [7],[8]
**Necessity**. Assume that $B(c^0, r) \subset D$. Take any point $y \in B(c^0, r)$. Then it can be represented as $y = c^0 + rh,\ \ h \in R^n,\ \ \|h\| \leq 1$. Since $y \in D$ each each defining inequality of $D$ must be satisfied

$$\langle a^i, y\rangle \leq b_i,\ \ i = 1, \dots, m.$$

Substituting $y = c^0 + rh$, we obtain

$$\langle a^i, c^0\rangle + r\langle a^i, h\rangle \leq b_i,\ \ \|h\| \leq 1,\ \ i = 1, \dots, m.$$

Because $\langle a^i, h\rangle \leq \|a^i\|\|h\| \leq \|a^i\|$, this leads to

$$\langle a^i, c^0\rangle + r\|a^i\| \leq b_i,\ \ i = 1, \dots, m.$$

**Sufficiency**. Conversely, suppose the inequalities

$$\langle a^i, c^0\rangle + r\|a^i\| \leq b_i,\ \ i = 1, \dots, m.$$

holds. Let $y \in B(c^0, r)$, so that $y = c^0 + rh$ with $\|h\| \leq 1$. Then follows

$$\langle a^i, y\rangle = \langle a^i, c^0\rangle + r\langle a^i, h\rangle,\ \ i = 1, \dots, m.$$

Since $\langle a^i, h\rangle \leq \|a^i\|\|h\| \leq \|a^i\|$, it follows that

$$\langle a^i, y\rangle \leq \langle a^i, c^0\rangle + r\|a^i\| \leq b_i,\ \ i = 1, \dots, m.$$

Thus all inequalities defining $D$ are satisfied, meaning $y \in D$. Therefore $B(c^0, r) \subset D$. □

Denote by $u^1, u^2 \dots u^k$ are centers of the spheres. Let $r_1, r_1 \dots r_k$ be their corresponding radii, respectively. Optimization problem of maximizing a total volume of $k$ non-overlapping spheres including $D$ is [8]:

$$\max_{(u,r)}\ V = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \sum_{i=1}^{k} r_i^n, \tag{1}$$

subject to:

$$\langle a^i, u^j\rangle + r_j\|a^i\| \leq b_i,\ \ i = 1, \dots, m,\ \ j = 1, \dots, k, \tag{2}$$

$$\|u^i - u^j\|^2 \geq (r_i + r_j)^2,\ \ i, j = 1, \dots, k,\ \ i < j, \tag{3}$$

$$r_i \geq 0,\ \ i = 1, \dots, k. \tag{4}$$

here $\Gamma$ is Euler's gamma function.
If $r_i = r,\ \ i = 1, \dots, k$ then problem (1)-(4) reduce to a classical sphere packing problem:

$$\max_{(u,r)}\ V = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} kr^n, \tag{5}$$

subject to:

$$\langle a^i, u^j\rangle + r\|a^i\| \leq b_i,\ \ i = 1, \ldots, m,\ \ j = 1, \ldots, k, \tag{6}$$

$$\|u^i - u^j\|^2 \geq 4r^2,\ \ i, j = 1, \ldots, k,\ \ i < j, \tag{7}$$

$$r \geq 0. \tag{8}$$

## 2. DC programming

Packing problems are inherently nonconvex due to the nonlinear and non-overlapping constraints between objects.
Let $x \in D \subset R^n$. A general DC optimization problem is given by:

$$(P)\ \begin{cases} f_0(x) = g_0(x) - h_0(x) \downarrow \min\limits_{x \in D}, \\ f_i(x) = g_i(x) - h_i(x) \leq 0, \quad i \in I_1 = \{1, \ldots, m\}. \\ f_j(x) = g_j(x) - h_j(x) = 0, \quad j \in I_1 = \{m+1, \ldots, l\}. \end{cases} \tag{9}$$

Here, $g_i(x), h_i(x)$, $i = 1 \ldots l$ are convex functions in $D$.
Further, suppose that the feasible set $S$ of Problem $(P)$ is nonempty:

$$S = \{x \in D \,|\, f_i(x) \leq 0,\ i \in I_1, f_j(x) \leq 0,\ j \in I_2\} \neq \emptyset. \tag{10}$$

Also assume that the optimal value of Problem $(\mathcal{P})$ is finite:

$$\Upsilon(P) = \inf_{x \in S} \{f_0(x)\} > -\infty. \tag{11}$$

Assume that a feasible starting point $x^0 \in D$ is given. And after several iterations, it has been produced sequences: $x^k \in D, k \in Z_+ = \{0, 1, 2, \ldots\}$. Then consider the following linearized problems [9]:

$$(P_k)\ \begin{cases} f_{0k}(x) = g_0(x) - \langle \nabla h_0(x^k), x\rangle - h_0(x) \downarrow \min\limits_{x \in D}, \\ f_{ik}(x) = g_i(x) - \langle \nabla h_i(x^k), x\rangle - h_i(x) \leq 0,\ \ i \in I_1 = \{1, \ldots, m\}. \\ f_{jk}(x) = g_j(x) - \langle \nabla h_j(x^k), x\rangle - h_j(x) = 0,\ \ j \in I_2 = \{m+1, \ldots, l\}. \end{cases} \tag{12}$$

This is main idea of DC Local search method to finding a local minimum to approximate of $f_0(x)$ around a current point $x^k$ using linearization of $h_i(x)$, where $\nabla h(x^k)$ is a gradient (or sub-gradient) of $h(\cdot)$.
Linearized problem $(P_k)$ is convex, it as well as following feasible set is convex.

$$S_k = \{x \in D \,|\, f_{ik}(x) \leq 0,\ i \in I_1, f_{jk}(x) = 0,\ j \in I_2\}. \tag{13}$$

Therefore, every iteration $x^k \in S_k$ as an approximation solution of the linearized problem $(P_k)$, satisfied following inequality:

$$f_{0k}(x^{k+1}) = g_0(x^{k+1}) - \langle \nabla h_0(x^{k+1}), x\rangle - g_0(x^{k+1}) \leq \Upsilon(P_k) + \delta_k. \tag{14}$$

where $\Upsilon(P_k)$ is optimal value of problem $(P_k)$:

$$\Upsilon(P_k) = \inf_x \{f_{0k} | x \in S_k, f_{ik} \le 0, i \in I_1, f_{jk}(x) \le 0,\ j \in I_2\}. \tag{15}$$

and sequence $\{\delta_k\}$ satisfies following condition:

$$\sum_{k=0}^{\infty} \delta_k < +\infty. \tag{16}$$

**Theorem 2.1.** *[2, 6] Suppose the sequence $\{x^k\}$ generated by (14), (16) converges to some $x^* \in D$. Then, the limit point $x^*$ is a solution of problem $P$.*

***Proof.*** **Feasibility.** Define the convex set $S_k = \{x \in D \,|\, f_{ik}(x) \le 0,\ i \in I_1, f_{jk}(x) = 0,\ j \in I_2\}$. The iterate satisfies:

$$x^{k+1} \in S_k, \quad f_{0k}(x^{k+1}) \le \Upsilon(P_k) + \delta_k \quad \sum_{k=0}^{\infty} \delta_k < +\infty.$$

Since $x^{k+1} \in S_k$, by construction of $S_k$, we have,

$$f_{ik}(x^{k+1}) \le 0, i \in I_1, \quad f_{jk}(x^{k+1}) = 0,\ j \in I_2.$$

Using convexity of $h_l$, it follows that $f_l(x) \le f_{lk}(x)$ for all $l$. Hence, every limit point $x^*$ of $\{x^k\}$ satisfies:

$$f_i(x^*) \le 0, i \in I_1, \quad f_j(x^*) = 0,\ j \in I_2.$$

so $x^*$ is feasible.

**Optimality**. For any feasible $y$ near $x^*$, we have $y \in S_k$ for $k$, large, and therefore

$$\Upsilon(P_k) \le f_{0k}(y).$$

From condition (14) and the property $f_0(x) \le f_{0k}(x)$,

$$f_0(x^{k+1}) \le f_{0k}(x^{k+1}) \le \Upsilon(P_k) + \delta_k.$$

Letting $k \to \infty$ and using $\delta_k \to 0$, we obtain

$$f_0(x^*) \le f_0(y).$$

Thus, $x^*$ is a local minimum. □

Introduce a following penalized problem

$$F_\sigma(x) = f_0(x) + \sigma W(x) \downarrow \min_{x \in R}. \tag{17}$$

where $\sigma$ is penalty parameter and penalty function $W(x)$ is following equality:

$$W(x) = \max\{0, f_1(x), \dots, f_m(x)\} + \sum_{j \in I_2} |f_j(x)|. \tag{18}$$

If $F_\sigma(x) = f_0(x) + \sigma W(x) = G_\sigma(x) - H_\sigma(x)$, $G_\sigma(x), H_\sigma(x)$ are:

$$H_\sigma(x) = h_0(x) + \sigma(\sum_{i \in I_1} h_i(x) + \sum_{j \in I_2} (g_j(x) + h_j(x))). \tag{19}$$

and

$$G_\sigma(x) = g_0(x) + 2\sigma \sum_{j\in I_2} \max\{g_j(x), h_j(x)\} + \sigma \max\{\sum_{j\in I_1} h_j(x); g_i(x) + \sum_{i\in I_1}^{j\neq i} h_j(x), i \in I_1\}. \tag{20}$$

Using penalty function function $W$ problem $(P_k)$ is following form:

$$(PP_k) \begin{cases} F_{0k}(x) = F_{0\sigma_k}(x) = G_{0k}(x) - \langle \nabla H_{0k}(x^k), x\rangle - H_{0k}(x) \downarrow \min\limits_{x\in D}, \\ F_{ik}(x) = F_{i\sigma_k}(x) = G_{ik}(x) - \langle \nabla H_{ik}(x^k), x\rangle - H_{ik}(x) \le 0, \ \ i \in I_1. \\ F_{jk}(x) = F_{j\sigma_k}(x) = G_{jk}(x) - \langle \nabla H_{jk}(x^k), x\rangle - H_{jk}(x) = 0, \ \ j \in I_2. \end{cases} \tag{21}$$

here $\{\sigma_k\}$ is a sequence penalty parameters and $G_k(x) = G_{\sigma_k}(x), H_k(x) = H_{\sigma_k}(x)$. Let starting point $x^0 \in D$, the current iteration $x^k$ and penalty parameter $\sigma^k$ given. And then next iteration $x^{k+1} \in D$ is intended from the following condition:

$$F_{0k}(x^{k+1}) - \delta_k = G_{0k}(x^{k+1}) - \langle \nabla H_{0k}(x^k), x^{k+1}\rangle - H_{0k}(x^{k+1}) - \delta_k \le \Upsilon(PP_k). \tag{22}$$

where $\Upsilon(PP_k)$ is optimal value of problem $(PP_k)$:

$$\Upsilon(PP_k) = \inf_x\{F_{0k}|x \in D, F_{ik} \le 0, i \in I_1, F_{jk}(x) \le 0, \ j \in I_2\} \tag{23}$$

The algorithm of the DC Local search method consists of the following steps:

Algorithm LSM: Local search method

Step 1: $k = 0, x^k = \underline{x}, \sigma_k = \underline{\sigma}, \varepsilon_x, \varepsilon_F, \varepsilon_G$ and tolerance threshold sequence $\{w_k\}$ : $\lim_{k\to\infty} w_k = 0$ are given.
Step 2: Find approximate solution $x(\sigma_k)$ at the current solution problem $(PP_k)$ and $x(\sigma_k) \in \Upsilon(PP_k) + \delta_k$.
Step 3:If $W(x(\sigma_k)) \le w_k$, then $\overline{\sigma} = \sigma_k, x(\overline{\sigma}) = x(\sigma_k)$ and goto Step 7.
Step 4: $(W(x(\sigma_k)) > w_k)$ if $(F_{0k}(x^k) - F_{0k}(x(\sigma_k))) \ge \rho_1\sigma_k(W(x^k) - W(x(\sigma_k))), \rho_1 \in ]0,1[$ then set $\overline{\sigma} = \sigma_k, x(\overline{\sigma}) = x(\sigma_k)$ and goto Step 7.
Step 5: $\overline{\sigma} = \rho_2\sigma_k, \rho_2 \in ]2,10[$ and find approximate solution $x(\overline{\sigma})$ of the linearized the following problem:

$$(\overline{PP_k}) \begin{cases} F_{0\overline{\sigma}}(x) = G_{0k}(x) - \langle \nabla H_{0k}(x(\overline{\sigma})), x\rangle - H_{0k}(x) \downarrow \min\limits_{x\in D}, \\ F_{i\overline{\sigma}}(x) = G_{ik}(x) - \langle \nabla H_{ik}(x(\overline{\sigma})), x\rangle - H_{ik}(x) \le 0, \ \ i \in I_1, \\ F_{j\overline{\sigma}}(x) = G_{jk}(x) - \langle \nabla H_{jk}(x(\overline{\sigma})), x\rangle - H_{jk}(x) = 0, \ \ j \in I_2. \end{cases} \tag{24}$$

then $x(\overline{\sigma}) \in \Upsilon(\overline{PP_k}) + \delta_k$.
Step 6: If $\|x^k - x(\overline{\sigma})\| \le \varepsilon_x$ or $\|F_{0k}(x^k) - |F_{0k}(x(\overline{\sigma}))\| \le \varepsilon_F$ or $\|\nabla F_{0k}(x^k)\| \le \varepsilon_G$ then algorithm stops and $x^k$ is final solution.
Step 7: Set $\sigma_k = \overline{\sigma}, x(\sigma_k) = x(\overline{\sigma})$ and goto Step 2.
Step 8: Set $\sigma_{k+1} = \overline{\sigma}, x(\sigma_{k+1}) = x(\overline{\sigma}), k = k + 1$ and goto Step 1.

## 3. Reduction of sphere packing problem to DC optimization

The packing problem can be represented as follows to DC optimization:

$$\min_{x\in D} V(x) = g_0(x) - h_0(x), \tag{25}$$

subject to:

$$f_{ij}(x) = g_{ij}(x) - h_{ij}(x) \leq 0, \ \ i < j, \ \ i,j = 1\ldots,k. \tag{26}$$

Here is:

$$g_0(x) = 0, \qquad h_0(x) = \tfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \textstyle\sum_{i=1}^{k} r_i^n,$$

$$g_{ij}(x) = (r_i + r_j)^2, \quad h_{ij}(x) = \|u^i - u^j\|^2, \ \ i,j = 1\ldots,k.$$

and feasible set $D$ is:

$$D = \{x \in R^n | \langle a^i, x\rangle + r_j\|a^i\| \leq b_i, r_j \geq 0, \ \ i,j = 1\ldots,k\}.$$

## 4. The test problem and results

In the test problem, we consider the classical Sangaku problem, which consists of packing six equal circles into a rectangle of dimensions $1 \times 1.934798$,which was studied Hidetoshi Fukugawa[10]. This problem s is maximizing the packing density and formulated as follows:

$$f = \frac{6\pi}{1 \times 1.934798} r^2 \to \max, \tag{27}$$

subject to the constraints

$$0 \leq x_i \leq 1.934798 - r, \quad i = 1\ldots,6, \tag{28}$$

$$0 \leq y_i \leq 1 - r, \quad i = 1\ldots,6, \tag{29}$$

$$\|c^i - c^j\|^2 \geq 4r^2; \quad i < j; \quad i,j = 1\ldots,6, \tag{30}$$

$$r \geq 0. \tag{31}$$

where, $c^1(x_1,y_1), c^2(x_2,y_2), c^3(x_3,y_3), c^4(x_4,y_4), c^5(x_5,y_5), c^6(x_6,y_6)$ are the centers of six circles in a feasible set $D = \{(x_i, y_j) \in \mathbb{R}^2 \mid 0 \leq x_i \leq 1.934798, \ \ 0 \leq y_j \leq 1 \ \ i,j = 1\ldots,6\}$ and $r$ is radius of circles.
DC Local search method for problem (27)-(31) run from 10 following starting points:

$$x_0^1 \approx (0.2, 0.2, 0.2, 0.3, 0.7, 0.22, 0.7, 0.4, 0.22, 1.2, 0.6, 0.22, 1.6, 0.22, 0.22, 1.7, 0.7, 0.22),$$
$$x_0^2 \approx (0.2, 0.2, 0.22, 0.3, 0.7, 0.22, 0.7, 0.4, 0.2, 1.2, 0.22, 0.22, 1.3, 0.7, 0.22, 1.7, 0.4, 0.22),$$
$$x_0^3 \approx (1, 0.7, 0.23, 1.7, 0.26, 0.23, 0.4, 0.7, 0.23, 1.5, 0.7, 0.23, 0.3, 0.26, 0.23, 0.8, 0.3, 0.23),$$
$$x_0^4 \approx (0.2, 0.2, 0.23, 0.3, 0.7, 0.23, 0.7, 0.4, 0.23, 1.2, 0.6, 0.23, 1.6, 0.23, 0.23, 1.7, 0.7, 0.2),$$
$$x_0^5 \approx (0.2, 0.2, 0.24, 0.3, 0.7, 0.24, 0.7, 0.4, 0.24, 1.2, 0.24, 0.24, 1.3, 0.7, 0.2, 1.7, 0.4, 0.24),$$
$$x_0^6 \approx (1, 0.7, 0.24, 1.7, 0.26, 0.24, 0.4, 0.7, 0.24, 1.5, 0.7, 0.24, 0.3, 0.26, 0.24, 0.8, 0.3, 0.24).$$
$$x_0^7 \approx (0.2, 0.2, 0.25, 0.3, 0.7, 0.25, 0.7, 0.4, 0.2, 1.2, 0.6, 0.25, 1.6, 0.25, 0.25, 1.7, 0.7, 0.25),$$
$$x_0^8 \approx (0.2, 0.2, 0.25, 0.3, 0.7, 0.25, 0.7, 0.4, 0.25, 1.2, 0.25, 0.25, 1.3, 0.7, 0.2, 1.7, 0.4, 0.25),$$
$$x_0^9 \approx (1, 0.7, 0.26, 1.7, 0.26, 0.26, 0.4, 0.7, 0.26, 1.5, 0.7, 0.26, 0.3, 0.26, 0.26, 0.8, 0.3, 0.26),$$
$$x_0^{10} \approx (1, 0.7, 0.26, 1.7, 0.26, 0.26, 0.4, 0.7, 0.26, 1.5, 0.7, 0.26, 0.3, 0.26, 0.26, 1, 0.3, 0.26).$$

In table 1, $x_0$ denotes the initial point number, $f_0(x_0)$ is the value of the cost function at the starting point. Furthermore, $f_0(x^*)$ denotes the value of the cost function at the critical point $x^*$. The quantity PK is the number of solved linearized subproblems, with best solution value 0.7075.

**Table 1.** Solution of 6 equal circle problem

| $x_0$ | $f_0(x_0)$ | $f_0(x^*)$ | PK | times(s) |
|---|---|---|---|---|
| 1 | 0.471 | 0.7075 | 5 | 0.69 |
| 2 | 0.491 | 0.7075 | 5 | 1.04 |
| 3 | 0.515 | 0.7075 | 5 | 0.99 |
| 4 | 0.536 | 0.7075 | 5 | 0.95 |
| 5 | 0.561 | 0.7075 | 4 | 0.90 |
| 6 | 0.585 | 0.7075 | 4 | 0.83 |
| 7 | 0.608 | 0.7075 | 4 | 0.75 |
| 8 | 0.613 | 0.7075 | 3 | 0.69 |
| 9 | 0.619 | 0.7075 | 3 | 0.65 |
| 10 | 0.653 | 0.7075 | 3 | 0.60 |

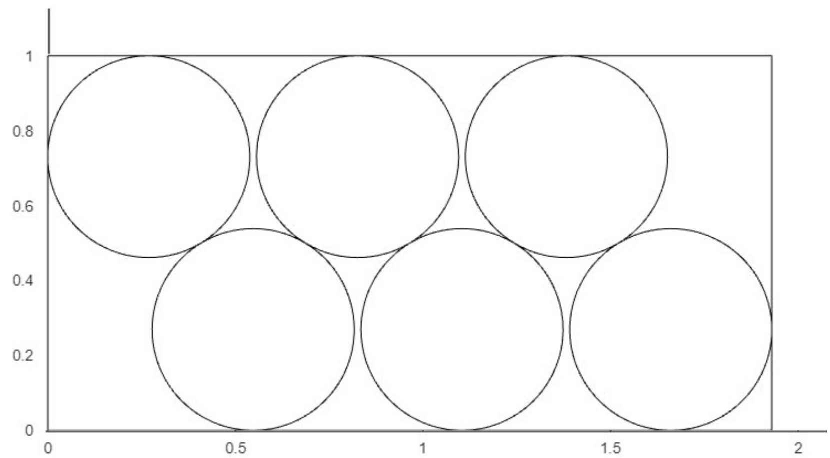


**Figure 1.** Best results for 6 equal circle problem

Now we consider the optimization problem of Sangaku 6 circles with different radii inside a bounded square domain. The problem is formulated as:

$$f = \frac{\pi}{1 \times 1.934798} \sum_{i=1}^{6} r_i^2 \to \max, \tag{32}$$

subject to the constraints

$$0 \leq x_i \leq 1.934798 - r_i; \quad i = 1\ldots, 6, \tag{33}$$

$$0 \leq y_i \leq 1 - r_i; \quad i = 1\ldots, 6, \tag{34}$$

$$\|c^i - c^j\|^2 \geq \|r_i + r_j\|^2; \quad i < j; \quad i,j = 1\ldots, 6, \tag{35}$$

$$r_i \geq 0; \quad i = 1\ldots, 6. \tag{36}$$

where, $c^1(x_1, y_1), c^2(x_2, y_2), c^3(x_3, y_3), c^4(x_4, y_4), c^5(x_5, y_5), c^6(x_6, y_6)$ are the centers of six circles in a feasible set $D$ and $r_1, r_2, \ldots, r_6$ are radii of circles.
DC Local search method for problem (32)-(36) run from 10 following starting points:

$$
\begin{aligned}
x_0^1 &\approx (0.5, 0.5, 0.5, 1.2, 0.2, 0.22, 1.08, 0.8, 0.16, 1.1, 0.55, 0.1, 1.2, 0.7, 0.03, 1.6, 0.6, 0.36),\\
x_0^2 &\approx (0.9, 0.7, 0.04, 0.9, 0.7, 0.01, 0.46, 0.5, 0.5, 1.4, 0.5, 0.5, 0.97, 0.72, 0.15, 0.9, 0.9, 0.1),\\
x_0^3 &\approx (1.05, 0.7, 0.02, 1, 0.7, 0.04, 1, 0.6, 0.02, 1.46, 0.5, 0.47, 0.5, 0.5, 0.5, 1.01, 0.9, 0.13),\\
x_0^4 &\approx (0.9, 0.9, 0.12, 1.46, 0.5, 0.47, 0.5, 0.5, 0.5, 1, 0.27, 0.01, 1.01, 0.13, 0.13, 1, 0.3, 0.04),\\
x_0^5 &\approx (0.9, 0.9, 0.1, 0.9, 0.3, 0.04, 1.43, 0.5, 0.5, 0.9, 0.13, 0.1, 0.46, 0.5, 0.46, 0.9, 0.4, 0.02),\\
x_0^6 &\approx (0.9, 0.9, 0.1, 0.9, 0.1, 0.1, 1.4, 0.5, 0.5, 0.94, 0.3, 0.04, 0.9, 0.7, 0.04, 0.46, 0.46, 0.46),\\
x_0^7 &\approx (1, 0.1, 0.1, 1, 0.69, 0.04, 1.2, 0.93, 0.07, 1.46, 0.467, 0.47, 0.5, 0.5, 0.5, 1, 0.87, 0.13).\\
x_0^8 &\approx (1, 0.1, 0.1, 1, 0.86, 0.13, 1.46, 0.46, 0.46, 0.5, 0.5, 0.5, 0.84, 0.94, 0.05, 1.2, 0.9, 0.06),\\
x_0^9 &\approx (0.5, 0.5, 0.5, 0.93, 1, 0.09, 1.14, 0.2, 0.2, 1.24, 0.72, 0.28, 1.64, 0.3, 0.3, 1.7, 0.73, 0.2),\\
x_0^{10} &\approx (0.5, 0.5, 0.5, 1.5, 0.5, 0.47, 1.8, 0.08, 0.07, 1.02, 0.866, 0.14, 1.8, 0.9, 0.1, 1, 0.1, 0.1).
\end{aligned}
$$

In table 2, $x_0$ denotes the initial point number, $f_0(x_0)$ is the value of the cost function at the starting point. Furthermore, $f_0(x^*)$ denotes the value of the cost function at the critical point $x^*$. The quantity PK is the number of solved linearized subproblems, with best solution value 0.84.

**Table 2.** Solution of 6 inequal circle problem

| $x_0$ | $f_0(x_0)$ | $f_0(x^*)$ | PK | times(s) |
|---|---|---|---|---|
| 1 | 0.7631 | 0.84 | 5 | 0.91 |
| 2 | 0.7938 | 0.84 | 5 | 0.83 |
| 3 | 0.7941 | 0.84 | 5 | 0.85 |
| 4 | 0.8171 | 0.84 | 4 | 0.76 |
| 5 | 0.8174 | 0.84 | 4 | 0.72 |
| 6 | 0.8193 | 0.84 | 4 | 0.79 |
| 7 | 0.8239 | 0.84 | 3 | 0.61 |
| 8 | 0.8266 | 0.84 | 3 | 0.53 |
| 9 | 0.828 | 0.84 | 3 | 0.57 |
| 10 | 0.839 | 0.84 | 3 | 0.51 |

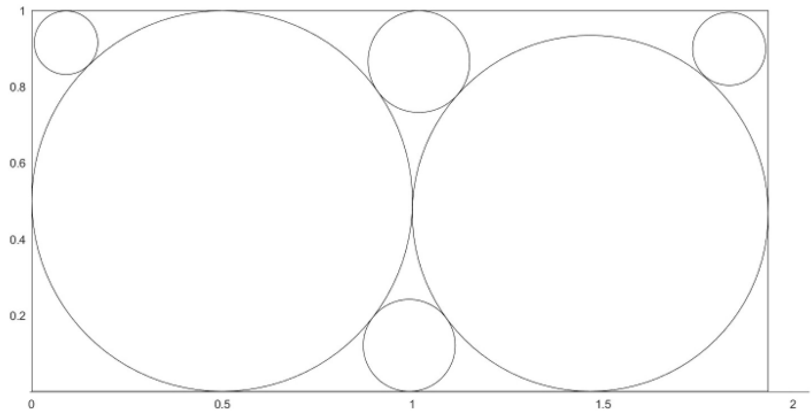


**Figure 2.** Best results for 6 inequal circle problem

## 5. Conclusion

In this paper, the sphere packing problem, which non-convex optimization problem was reformulated by a DC (Difference of Convex functions) programming and the DC constraints. Based on DC Local Search Method, we attempt to find the best solution by iteration. In experiments, we choose Sangaku problem of packing 6 circles in rectangle of $1 \times 1.934798$ size. Computational results are performed by Gekko package, Python Jupyter Notebook

## Declarations

**Ethics approval and consent to participate**. Not applicable.
**Data availability statement**. No new data were created or analyzed in this study. Therefore, data sharing is not applicable to this article.
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